# Practice Exam 2 solutions

1) let $P, Q$ be statements. prove the following are always true (a) $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$

(b) $[P \Rightarrow (Q \wedge \neg Q)] \Rightarrow \neg P$

pf:
we write down the truth tables and use the facts, $T(P \Rightarrow Q) = 1 - T(P) + T(P)T(Q)$, $T(\neg P) = 1 - T(P)$
$T(P \wedge Q) = T(P)T(Q)$, $T(P \vee Q) = T(P) + T(Q) - T(P)T(Q)$

| $P$ | $Q$ | $P \Rightarrow Q$ | $P \wedge (P \Rightarrow Q)$ | (a) | $(Q \wedge \neg Q)$ | $P \Rightarrow (Q \wedge \neg Q)$ | (b) |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |

$\square$

2.) let $P, Q$ be statements. Prove $P, Q$ are prop. equiv. iff $P \Leftrightarrow Q$.

pf:
($\Rightarrow$) s'pose $P, Q$ prop. equiv. Then $T(P) = T(Q)$. consider $T(P \Leftrightarrow Q) = T(P \Rightarrow Q)T(Q \Rightarrow P)$
$= [1 - T(P) + T(P)T(Q)][1 - T(Q) + T(Q)T(P)] = \{1 - T(P) + T(P)^2\}^2$ but if $T(P) = 1$ then $T(P \Leftrightarrow Q) = 1$
if $T(P) = 0 \Rightarrow T(P \Leftrightarrow Q) = 1$ either way $T(P \Leftrightarrow Q)$ is always true.

($\Leftarrow$) now s'pose $T(P \Leftrightarrow Q)$ is always true ie $T(P \Leftrightarrow Q) = 1 \Rightarrow T(P \Rightarrow Q) = 1$ and $T(Q \Rightarrow P) = 1$
or $\begin{cases} 1 - T(P) + T(P)T(Q) = 1 \\ 1 - T(Q) + T(P)T(Q) = 1 \end{cases} \Rightarrow \begin{cases} T(P) = T(P)T(Q) \\ T(Q) = T(P)T(Q) \end{cases}$ if $T(P) = 0 \Rightarrow T(Q) = 0$ if $T(P) = 1 \Rightarrow T(Q) = 1$
so $T(P) = T(Q)$ $\square$

3.) Prove $\sqrt{10}$ is irrational.

pf:
S'pose $\sqrt{10}$ rat'l. Then $\exists\, p, q \in \mathbb{Z}$, $q \neq 0$ s.t. $p, q$ have no common terms and $\sqrt{10} = \frac{p}{q}$
$\Rightarrow 10 = \frac{p^2}{q^2} \Rightarrow p^2 = 10q^2$ consider mod by 10 $\Rightarrow p^2 \bmod 10 = 0 \Rightarrow (p \bmod 10)^2 = 0$
since mod arithmetic preserves operations. $\Rightarrow p \bmod 10 = 0$ so $10 | p \Rightarrow p = 10t$ for $t \in \mathbb{Z}$
$\Rightarrow 10^2 t^2 = 10q^2 \Rightarrow q^2 = 10t^2$ so a similar argument shows $10 | q \Rightarrow p, q$ have common factor.
contradiction $\Rightarrow \sqrt{10}$ is irrational. $\square$

4.) Prove $\exists$ irrational #'s $x, y$ s.t. $x^y$ is rational.

pf:
consider $\sqrt{2}$, we know $\sqrt{2}$ is irrational. then consider $\sqrt{2}^{\sqrt{2}}$ if $\sqrt{2}^{\sqrt{2}}$ is rational done
take $x = y = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational take $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$ then
$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ which is rational. $\square$

5.) Prove $\forall n \in \mathbb{N}$ : $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

pf:
we prove via induction on $n$. 1st base case: $n=0$ so $LHS = (x+y)^0 = 1$ the $RHS = \sum_{k=0}^{0} \binom{0}{k} x^{-k} y^k$ but $\sum_{k=0}^{0}$ forces $k=0$ so $RHS = \binom{0}{0} x^0 y^0 = 1$. next we do the inductive step. So suppose true for $n$ and we show for $n+1$.

Then $(x+y)^{n+1} = (x+y)(x+y)^n = (x+y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$

via the assumption. Then we resum the 2nd sum, $j = k+1$ and just replace $j$ w/ $k$ in 1st sum

$\Rightarrow (x+y)^{n+1} = \sum_{j=0}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n+1} \binom{n}{j-1} x^{n-(j-1)} y^j = \sum_{j=0}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n+1} \binom{n}{j-1} x^{n+1-j} y^j$

$= \binom{n}{0} x^{n+1} + \sum_{j=1}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n} \binom{n}{j-1} x^{n+1-j} y^j + \binom{n}{n} y^{n+1} = \binom{n}{0} x^{n+1} + \sum_{j=1}^{n} \binom{n+1}{j} x^{n+1-j} y^j + \binom{n}{n} y^{n+1}$

via def of the binomial coefficients. but $\binom{n}{0} = \binom{n}{n} = 1 = \binom{n+1}{0} = \binom{n+1}{n+1}$ so

$(x+y)^{n+1} = \binom{n+1}{0} x^{n+1} + \sum_{j=1}^{n} \binom{n+1}{j} x^{n+1-j} y^j + \binom{n+1}{n+1} y^{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} x^{n+1-j} y^j$ which shows $n+1$ case

so via the induction hypothesis it's true for all $n \in \mathbb{N}$. $\square$

6.) let $n \in \mathbb{N}$. Prove $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

pf:
we prove via induction on $n$. 1st is the base case: $n=0$. $LHS = \sum_{k=0}^{0} k^2 = 0^2 = 0$ and $RHS = \frac{0(0+1)(0+1)}{6} = 0$ so $LHS = RHS$. next the inductive step. suppose true for $n$ and we demonstrate $n+1$. then $\sum_{k=0}^{n+1} k^2 = (n+1)^2 + \sum_{k=0}^{n} k^2 = (n+1)^2 + \frac{n(n+1)(2n+1)}{6}$ via assumption

Then $\sum_{k=0}^{n+1} k^2 = \frac{6(n+1)^2 + n(n+1)(2n+1)}{6} = \frac{n+1}{6}\left(6(n+1) + n(2n+1)\right) = \frac{n+1}{6}\left(6n+6 + 2n^2 + n\right) = \frac{n+1}{6}(2n^2 + 7n + 6)$

$= \frac{n+1}{6}(n+2)(2n+3) = \frac{(n+1)([n+1]+1)(2[n+1]+1)}{6}$ so true for $n+1$ case. Thus via the induction hypothesis, true for all $n \in \mathbb{N}$ $\square$.

7.) let $f, g$ be fncs on $I \setminus \{a\}$ w/ $a \in I$, s.t. $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$. Prove $\lim_{x \to a} (f(x) g(x)) = LM$.

pf:
$\forall \varepsilon > 0$, let $\delta = \min\{\delta_1, \delta_2\}$ s.t. if $|x-a| < \delta_1$ then $|f(x) - L| < \frac{\varepsilon}{2M}$, and if $|x-a| < \delta_2$ then $|g(x) - M| < \frac{\varepsilon}{2(|L|+1)}$ then if $|x-a| < \delta$ consider $|f(x) g(x) - LM|$

$= |f(x)g(x) - f(x)M + f(x)M - LM| = |f(x)(g(x) - M) + M(f(x) - L)| \le |f(x)||g(x) - M| + M|f(x) - L|$

via the triangle $\le$, since $f(x)$ has limit $L$ at $x=a$ then $|f(x)| \le |L| + 1$ so

$|f(x) g(x) - LM| \le (|L|+1)|g(x) - M| + M|f(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$.

8.) let $f, g$ be fncs on $I \setminus \{a\}$ w/ $a \in I$ s.t. $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Prove $\lim_{x \to a} (f(x) + g(x)) = L + M$.

pf:
$\forall \varepsilon > 0$, let $\delta = \min\{\delta_1, \delta_2\}$ s.t. if $|x-a| < \delta_1$ then $|f(x) - L| < \frac{\varepsilon}{2}$ and if $|x-a| < \delta_2$ then $|g(x) - M| < \frac{\varepsilon}{2}$ so if $|x-a| < \delta$ consider $|f(x) + g(x) - (L+M)|$

$= |(f(x) - L) + (g(x) - M)| \le |f(x) - L| + |g(x) - M|$ via the triangle $\le$, but $|f(x) - L| < \frac{\varepsilon}{2}$

and $|g(x) - M| < \frac{\varepsilon}{2}$ so $|f(x) + g(x) - (L+M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\square$.

9.) Define $H(x) = \begin{cases} 5 & x < 0 \\ 0 & x \ge 0 \end{cases}$ Prove $H(x)$ doesn't have a limit at $x = 0$.

pf:
S'pose $H(x)$ has a limit $L$ at $x = 0$. Then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $|x| < \delta \Rightarrow |H(x) - L| < \varepsilon$

Moreover if $\lim_{x \to 0} H(x) = L$ then $\underset{(1)}{\lim_{x \to 0^-} H(x) = L}$ and $\underset{(2)}{\lim_{x \to 0^+} H(x) = L}$. but (1) implies

if $x \ge 0$ and $\delta > 0$ then $|H(x) - 0| < \frac{1}{2}$ so the triangle $\le \Rightarrow |L| < \frac{1}{2}$ here we picked $\varepsilon = \frac{1}{2}$

similarly (2) if $x < 0$ and $\delta > 0$ then $|H(x) - 5| < \frac{1}{2}$ and triangle $\le \Rightarrow |5 - L| < \frac{1}{2}$

so using the triangle $\le$ one more time $\Rightarrow |5| = |5 - L + L| \le |5 - L| + |L| < \frac{1}{2} + \frac{1}{2} = 1$

or $|5| < 1$ which is nonsense. contradiction $\Rightarrow H(x)$ has no limit at $x = 0$. $\square$